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A FEW REMARKS ON COLLET-ECKMANN ATTRACTORS, LYAPUNOV ATTRACTORS AND ASYMPTOTICALLY STABLE ATTRACTORS

Jaeyoo Choy*

Abstract. In this short note, we explicate a relation among Collet-Eckmann attractors, Lyapunov attractors and asymptotically stable attractors in the sense of J. Milnor[4].

1. Introduction

This note is intended to have some remarks on attractors, whose necessity arises from dynamics on non-compact spaces. The examples of those motivations are ubiquitous: Hénon attractors, Rössler attractors, Lorenz attractors, Tamari attractors (ref. [6]); Ginzburg-Landau equations, Navier-Stokes equations in differential equation systems as global attractors.

In Milnor's work[4], he studies various existing definitions of attractors on a compact topological space. As was pointed out in [4], however, some definitions are too restrictive to omit certain interesting attractors. Attractors defined on compact metric spaces are one of such. Nonetheless, the definitions in [4] can be obviously generalized to non-compact spaces by simply ignoring the assumption: the base metric space is compact. It causes some problems for this heedless generalization (cf. [1]). This can be partially avoided by taking the one-point compactification. But, it again causes a problem regarding minimality and maximality. So, as was suggested by Milnor in [4, §1] (cf. [7]), we let those canonical definitions be left, and impose a canonical assumption: an attractor with compact boundary, practically arising from many examples.

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Motivated by the above discussion, we try to identify the attractors of three kinds: Collet-Eckmann attractors, Lyapunov attractors, asymptotically stable attractors, under the assumption. Most naive definition is the Collet-Eckmann attractors, while the most restrictive one is the asymptotically stable attractors ([4, p.179][7, p.786]). Our aim, following the teniques in [5], is to show the naive one becomes the restrictive one when the boundary is compact. A few steps away from the technicalities of the proof and the abundance of the examples, an algebraic example finding roots of polynomials is an addendum of our note, which is entirely from the work of McMullen[3].

2. Collet-Eckmann attractors and asymptotically stable attractors

Let f be a continuous map from a locally compact metric space X to itself. The map $f^n : X \to X$ denotes the n-th iteration of f.

Definition 2.1. For a non-empty proper closed subset $A \subset X$, A is a *Collet-Eckmann attractor* (named followed by Milnor's paper[4, p.179]) if

1. $f(A) = A$
2. there exists a neighborhood U of A such that

$$\bigcap_{n \in \mathbb{Z}_+} f^n(U) = A.$$

Definition 2.2. For a non-empty proper closed subset $A \subset X$, A is a *Lyapunov attractor* (ref. [5, p.517]) if

1. it is a Collet-Eckmann attractor
2. there exists a sufficiently small neighborhood V of A such that $f(V) \subset V$.

Definition 2.3. For a non-empty proper closed subset $A \subset X$, A is an *asymptotically stable attractor* (ref. [5, p.517]) if

1. $f(A) = A$
2. there exists a neighborhood U of A such that for any given neighborhood V of A, $f^n(U) \subset V$ for some $n \in \mathbb{Z}_+$.

We have the direct implications among the above attractors: Asymptotically stable attractor $\Rightarrow$ Lyanpunov attractor $\Rightarrow$ Collet-Eckmann attractor.

THEOREM 2.4. *Let X be any locally compact metric space. If A is a Collet-Eckmann attractor with the compact boundary, then A is an asymptotically stable attractor.*

Proof. Let A be a Collet-Eckmann attractor and U be an open neighborhood of A in Definition 2.1, i.e. $\bigcap_{n\in\mathbb{Z}_+} f^n(U) = A$. Since the boundary ∂A is compact, we can set U as a union of A and an open neighborhood U' of ∂A such that U' has the compact closure $\overline{U}'$. Indeed, $U = A\cup U' = A^o \cup U'$ where A^o is the interior of A, thus U is an open neighborhood of A. Further, by replacing U into a smaller neighborhood, we have $\bigcap_{n\in\mathbb{Z}_+} f^n(\overline{U}) = A$.

Let us define an open set

$$U_n = \{x \in X | f^i(x) \in U \text{ for all } 0 \le i \le n\}.$$

Then, $U = U_0 \supset U_1 \supset U_2 \supset \cdots \supset A$ and $f(U_n) \subset U_{n-1}$. Let

$$W = \bigcap_{n\in\mathbb{Z}_+} U_n.$$

Claim that $W = U_n$ for some $n > 0$, and hence that W is an open set. Otherwise, we have $x_n \in U_n - U_{n+1}$. Note that x_n does not lie in A, but does in U'. Therefore the same things hold for $y_n = f^n(x_n)$. Since $\overline{U}'$ is compact and each $y_n \in \overline{U}'$ (otherwise, $y_n \in A$, thus $x_n \in U_m$ for all $m > n$), a subsequence of $\{y_n\}$ has a limit y in $\overline{U}'$. Because $y_n \in \bigcap_{1\le i\le n} f^i(\overline{U}')$, the limit $y \in \bigcap_{i\in\mathbb{Z}_+} f^i(\overline{U}')$ (by the nested compact set theorem). Thus, y should be contained in A, which implies that $f(y) \in A$. However, $f(y_n) = f^{n+1}(x_n) \notin U$ (if so, $x_n \in U_{n+1}$). Thus, $f(y) \notin U$ which contradicts $f(y) \in A$. The claim is proven.

As a consequence of the claim, we have $f(W) \subset W \subset U$ and $\bigcap_{i\in\mathbb{Z}_+} f^i(W) = \bigcap_{i\in\mathbb{Z}_+} f^i(\overline{W}) = A$. Let us observe that $W - A^o \subset U'$ has the compact closure. Therefore, for any open neighborhood V of A, there exists $n > 0$ such that $f^n(W) \subset V$. This completes the proof. □

COROLLARY 2.5. *For a non-empty proper closed subset $A \subset X$ with the compact boundary, it is a Collet-Eckmann attractor iff a Lyapunov attractor iff an asymptotically stable attractor.* □

REMARK 2.6. (after McMullen[3]) As an incardination of Julia sets of rational functions on $\mathbb{C}$, there exists or may not exist an iterative algorithm finding roots of a given polynomial p, purely in terms of the coefficients of p. Newton's method, if it works, is a positive part of the existence of the algorithm, while in contrast, the nonexistence for a higher degree is shown by McMullen. An algorithm here is an assignment to p a rational function $T_p(z)$ in the coefficients of p. More precisely, he shows that if V is a non-empty open, connected subset of the set of monic polynomials in degree $d \geq 4$ (in one variable) consisting of polynomials such that the monodromy group G of $p \in V$ along closed curves in V induces a transitive action on the roots of p, and the image of G in the mapping class group of $\mathbb{C}$ minus the roots is an irreducible infinite group, then any algorithm $p \mapsto T_p$ on V fails to be an root-finding algorithm. (The proof stems on a preceding work on the modular group of punctured Riemann spheres.) This suggests that any rational algorithm, generically, does not admit a finite attractor, but the attractors lie in a bounded filled Julia set. Thus, an algorithm as Newton's method, generically have attractors with the compact boundary. In the case, detection of attractors of asymptotic stability or Lyapunov is not easy, but that of Collet-Eckmann is rather easy; Theorem 2.4 enters here.

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Department of Mathematics
Kyungpook National University
Daegu 702-701, Republic of Korea
E-mail: donvosco@knu.ac.kr