JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 3, September 2008

STRONG CONVERGENCE THEOREM FOR UNIFORMLY L-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove strong convergence theorems for a finite family of uniformly L-Lipschitzian mappings by a cyclic iterative algorithm in the framework of Banach spaces. Our results improve and extend the recent ones announced by many others.

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, K is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x,\ f\rangle = ||x||^2 = ||f||^2,\ ||f|| = ||x||\},\ \ \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. Let $T : K \to K$ be a mapping.

(1) T is said to be *uniformly $L-$Lipschitzian* if there exists $L > 0$ such that, for any $x, y \in K$,

$$||T^n x - T^n y|| \le L||x - y||,\ \ \forall n \ge 1;$$

(2) T is said to be *nonexpansive* if the following inequality holds for all $x, y \in K$

$$||Tx - Ty|| \le ||x - y||;$$

(3) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for any given

Received November 18, 2007; Accepted May 20, 2008.
2000 Mathematics Subject Classification: Primary 47H09; Secondary 47H10.
Key words and phrases: asymptotically pseudo-contractive mapping, normalized duality mapping, uniformly L-Lipschitzian mapping, Banach space.

$x,\ y \in K$,

$$||T^n x - T^n y|| \le k_n ||x - y||, \ \ \forall n \ge 1;$$

(4) T is said to be *asymptotically pseudo-contractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that, for any $x,\ y \in K$, there exists $j(x-y) \in J(x-y)$

$$\langle T^n x - T^n y,\ j(x-y)\rangle \le k_n ||x-y||^2, \ \ \forall n \ge 1.$$

REMARK 1.1. (1) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly $L-$Lipschitzian mapping, where $L = \sup_{n\ge 1}\{k_n\}$. And every asymptotically nonexpansive mapping is asymptotically pseudo-contractive, but the inverse is not true, in general.

(2) The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4]. Asymptotically nonexpansive mappings include nonexpansive mappings as a special case when $\{k_n\} = 1$.

Recently, Schu [7] proved the following strong convergence theorem for asymptotically pseudo-contractive mappings

THEOREM 1.1. *Let H be a Hilbert space, $\emptyset \ne K \subset H$ closed bounded convex; $L > 0$; $T : K \to K$ completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$ for all $n \ge 1$; $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$; $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$; $\epsilon \le \alpha_n \le \beta_n \le b$ for all $n \ge 1$ some $\epsilon > 0$ and some $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$; $x_1 \in K$; for all $n \ge 1$, define*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n. \end{cases}$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

In [1], Chang extended Theorem 1.3 to a real uniformly smooth Banach space. To be more precise, he proved the following theorem:

THEOREM 1.2. *Let E be a uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, $T : D \to D$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and $F(T) \ne \emptyset$, where $F(T)$ is the set of fixed points of T in D. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be for sequence in $[0, 1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \le 1$, $\beta_n + \delta_n \le 1$;
(b) $\alpha_n \to 0$, $\beta_n \to 0$ $(n \to \infty)$;

(c) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n = \infty$.

Let $x_0 \in D$ be any given point and let $\{x_n\}$, $\{y_n\}$ be the modified Ishikawa iterative sequence errors defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_m)x_n + \alpha_n T^n y_n + \gamma_n u_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \quad n \geq 0. \end{cases}$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*) \rangle \leq k_n \|x_n - x^*\|^2 - \phi(\|x_n - x^*\|), \quad \forall n \geq 0,$$

where $x^* \in F(T)$ is some fixed point of T in D, then $x_n \to x^*$ as $n \to \infty$.

The purpose of this paper is, by using a simple and quite different method, to prove some strong convergence theorems for a finite family of L-Lipschitzian mappings in stead of the assumption that T is a uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping in a Banach space. Our results extend and improve some recent results announced by Chang [1], Cho et al. [3], Ofoedu [6], Schu [7] and many others

In order to prove our main results, we also need followings lemmas:

LEMMA 1.3 ([2]). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$,*

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ j(x + y)\rangle, \ \ \forall j(x + y) \in J(x + y).$$

LEMMA 1.4 ([5]). *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying the following conditions:*

$$0 \leq \lambda_n \leq 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \ \ \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative number such that $\sigma_n = \circ(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

LEMMA 1.5. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + \lambda_n)a_n + b_n, \ \ \forall n \geq n_0,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2. Main results

THEOREM 2.1. *Let E be a real Banach space, K a nonempty closed convex subset of E and $\{T_i\}_{i=1}^N : K \to K$ a finite family of uniformly L-Lipschitzian mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i)$ denotes the set of fixed points of T_i. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ as $n \to \infty$. Let $\{\alpha_n\} \subset (0,1)$ be a sequence satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(c) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{s_n}^n x_n, \quad n \geq 0, \tag{2.1}$$

where $s_n = n \bmod N$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$. If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, $i = 1, 2, \ldots, N$, then $\{x_n\}$ converges strongly to a point $x^* \in F$.

Proof. First, we prove that the sequence $\{x_n\}$ defined by (2.1) is bounded. Actually, it follows from (2.1) and Lemma 1.3 that

$$\begin{aligned} &||x_{n+1} - x^*||^2 \\ &= ||(1-\alpha_n)(x_n - x^*) + \alpha_n(T_{s_n}^n x_n - x^*)||^2 \\ &\leq \|(1-\alpha_n)(x_n - x^*)\|^2 + 2\alpha_n\langle T_{s_n}^n x_n - x^*,\ j(x_{n+1} - x^*)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n L\|x_n - x_{n+1}\|\|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n\{k_n\|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\}. \end{aligned} \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\alpha_n)x_n + \alpha_n T_{s_n}^n x_n - x_n\| \\ &= \alpha_n\|x_n - T_{s_n}^n x_n\| \\ &\leq \alpha_n\|x_n - x^*\| + \alpha_n\|T_{s_n}^n x_n - x^*\| \\ &\leq \alpha_n(1+L)\|x_n - x^*\|. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned}||x_{n+1}-x^*||^2 &\le (1-\alpha_n)^2||x_n-x^*||^2\\ &\quad + 2\alpha_n\{k_n||x_{n+1}-x^*||^2-\phi(||x_{n+1}-x^*||)\}\\ &\quad + 2L\alpha_n^2(1+L)\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\le (1-\alpha_n)^2||x_n-x^*||^2+2\alpha_n\{k_n||x_{n+1}-x^*||^2\\ &\quad -\phi(||x_{n+1}-x^*||)\}+L\alpha_n^2(1+L)(\|x_n-x^*\|^2\\ &\quad +\|x_{n+1}-x^*\|^2).\end{aligned}\tag{2.4}$$

Therefore, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n)^2+L\alpha_n^2(1+L)}{1-2\alpha_nk_n-\alpha_n^2L(1+L)}\|x_n-x^*\|^2\\ &\quad -\frac{2\alpha_n\phi(||x_{n+1}-x^*||)}{1-2\alpha_nk_n-\alpha_n^2L(1+L)}\\ &= [1+\frac{\alpha_n^2+2\alpha_n(k_n-1)+2\alpha_n^2L(1+L)}{1-2\alpha_nk_n-\alpha_n^2L(1+L)}]\|x_n-x^*\|^2\\ &\quad -\frac{2\alpha_n\phi(||x_{n+1}-x^*||)}{1-2\alpha_nk_n-\alpha_n^2L(1+L)}.\end{aligned}\tag{2.5}$$

Since $\alpha_n\to 0$ as $n\to\infty$, there exists a positive integer n_0 such that

$$\frac{1}{2}<1-2\alpha_nk_n-\alpha_n^2L(1+L)\le 1,\quad \forall n\ge n_0.$$

Therefore, it follows from (2.5) that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\le [1+2(\alpha_n^2+2\alpha_n(k_n-1)+2\alpha_n^2L(1+L))]\|x_n-x^*\|^2\\ &\quad -2\alpha_n\phi(||x_{n+1}-x^*||)\\ &\le [1+2(\alpha_n^2+2\alpha_n(k_n-1)+2\alpha_n^2L(1+L))]\|x_n-x^*\|^2.\end{aligned}\tag{2.6}$$

By the conditions (b) and (c), we have

$$2\sum_{n=0}^{\infty}[\alpha_n^2+2\alpha_n(k_n-1)+2\alpha_n^2L(1+L)]<\infty.$$

It follows from Lemma 1.5 that the limit $\lim_{n\to\infty}||x_n-x^*||$ exists. Therefore, the sequence $\{||x_n-x^*||\}$ is bounded. Without loss of generality, we can assume that $||x_n-x^*||^2\le M_1$, where M_1 is an appropriate positive constant. Take $\theta_n=||x_n-x^*||$, $\lambda_n=2\alpha_n$ and

$$\sigma_n=2[(\alpha_n^2+2\alpha_n(k_n-1)+2\alpha_n^2L(1+L))]M_1.$$

(2.6) can be written as

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \quad \forall n \geq n_0.$$

By the conditions (a)-(c), we know that all the conditions in Lemma 1.4 are satisfied. Therefore, it follows that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0,$$

that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 2.1. Theorem 3.1 mainly improves the corresponding ones announced by Chang [1], Cho et al. [3]. Ofoedu [6] and Schu [7]. The method given in the proof of Theorem 2.1 is also different from the method given by Ofoedu [6].

Remark 2.2. Under suitable conditions, the sequence $\{x_n\}$ defined by (2.1) in Theorem 2.1 can be generalized to the iterative sequences with errors. Since the proof is trivial, we omit it here.

As a corollary of Theorem 2.1, we have the following result immediately.

Corollary 2.2. *Let E be a real Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a uniformly L-Lipschitzian mapping with $F(T) \neq \emptyset$, where $F(T)$ denotes the set of fixed points of T. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$ as $n \to \infty$. Let $\{\alpha_n\} \subset [0,1]$ be a sequence satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(c) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, \, j(x - x^*) \rangle \leq k_n ||x - x^*||^2 - \phi(||x - x^*||)$$

for all $j(x - x^*) \in J(x - x^*)$ and $x \in K$, then $\{x_n\}$ converges strongly to x^*.

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