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Almost Projective Modules over Artin Algebras

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ABSTRACT. The main result of this paper is a characterization of almost projective modules over artin algebras by means of irreducible maps and almost split sequences. A module X is an almost projective module if and only if it has a presentation $0 \longrightarrow L \overset{\alpha}{\longrightarrow} P \longrightarrow X \longrightarrow 0$ with projective module P and irreducible maps α. Let X be an injective almost projective non simple module and $0 \to \mathrm{Dtr}(x) \to E \to X \to 0$ be an almost split sequence. If $E = E_1 \oplus E_2$ is a direct decomposition of indecomposable modules then $\ell(X) = 3$.

Background and notations. The notion of almost projective modules was suggested by M. Auslander and R. M. Villa generalizes this concepts in [7]. Using the notion of an almost projective modules Villa proved that any non semisimple artin algebra has at least one indecomposable module A such that the almost split sequence $0 \to \mathrm{Dtr}(A) \to E \to A \to 0$, has E indecomposable. Also Villa showed that: if Λ is an artin algebra such that each indecomposable non projective module X has the property that, in each almost split sequence $0 \to \mathrm{Dtr}(X) \to E \to X \to 0$, E is indecomposable, then Λ is self-injective[7].

Here we only consider artin algebras and finitely generated modules. Let Λ be an artin algebra and mod Λ the category of finitely generated Λ-modules. If Λ is a right artin algebra, then every finitely generated Λ-module is uniquely to isomorphism a finite direct sum of indecomposable Λ-modules. If Λ is a right artin algebra and X is a finitely generated right Λ-module, then X is artinian and noetherian.

DEFINITION 1. An almost projective module X is a non projective module such that given a non split exact sequence $0 \to A \to B \to$

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$X \to 0$ and a proper submodule C of X, the pull back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & W & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle 1_A} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

satisfies that the exact sequence

$$0 \to A \to W \to C \to 0$$

splits. One has also the dual notion of an almost injective.

DEFINITION 2. An almost injective module X is a non injective module such that given a non split sequence $0 \to X \to B \to C \to 0$ and a proper factor A of X, the push out

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & W & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

satisfies that the exact sequence

$$0 \to A \to W \to C \to 0$$

splits.

It is shown in [**7**] that every self-injective algebra has no almost projective non simple modules.

DEFINITION 3(AUSLANDER AND REITEN, [**2**]). Let A and B be in mod Λ. A morphism $f : A \to B$ is called an irreducible morphism if f is neither a splittable epimorphism nor a splittable monomorphism but whenever the diagram

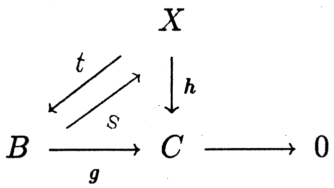
$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ g \searrow & & \nearrow h \\ & C & \end{array}$$

commutes either g is a splittable monomorphism or h is a splittable epimorphism.

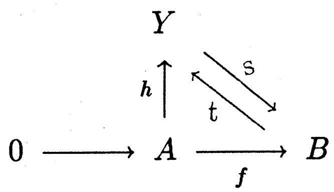
The connection between irreducible morphisms and short exact sequences of modules is given in the next proposition taken from [**2**].

PROPOSITION 4. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a non exact sequence in an arbitrary abelian category.*

(i) $f : A \to B$ *is irreducible if and only if the morphism* $g : B \to C$ *has the property: given any morphism* $h : X \to C$, *there is either a morphism* $t : X \to B$ *such that* $gt = h$ *or a morphism* $s : B \to X$ *such that* $hs = g$:



(ii) $g : B \to C$ *is irreducible morphism if and only if the morphism* $f : A \to B$ *has the property: given any morphism* $h : A \to Y$, *there is a morphism* $t : B \to Y$ *such that* $tf = h$ *or a morphism* $s : Y \to B$ *such that* $sh = f$:



We can characterize almost projective modules by means of irreducible maps. In [7], Villa proved the following theorem under the condition that X has no projective summands. However we show that the condition that X has no projective summands is not necessary.

THEOREM 5. *The following conditions are equivalent.*

(i) X *is an almost projective module.*

(ii) X *has a presentation* $0 \to L \xrightarrow{\alpha} P \to X \to 0$ *with projective module* P *and irreducible morphsim* α.

PROOF. (i) $\Rightarrow$ (ii): Suppose that X is an almost projective module. Then there is a free module P and an epimorphism $\beta : P \to X$. Let $L = \operatorname{Ker}\beta$. Then the sequence $0 \to L \xrightarrow{\alpha} P \xrightarrow{\beta} X \to 0$ is an exact sequence with the projective module P, since any free module is projective. Let $f : Y \to X$ be any map. First we suppose that the map $f : Y \to X$ is an epimorphism. Then there is a map $h : P \to Y$ such that $fh = \beta$, since P is projective. Hence by Proposition 4, α is irreducible. Second we suppose that the map $f : Y \to X$ is not an epimorphsim. Then $Z = \operatorname{Im} f$ is a proper submodule of X. Since X is an almost projective module, in the pull back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\phi} & W & \xrightarrow{\psi} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & L & \xrightarrow[\alpha]{} & P & \xrightarrow[\beta]{} & X & \longrightarrow & 0 \end{array}$$

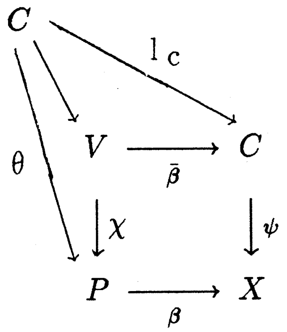
the sequence $0 \to L \to W \to Z \to 0$ splits. Hence there is a map $\gamma : Z \to W$ such that $\beta g \gamma = i$. Then there is a map $h : Y \to P$ such that $\beta h = f$. Hence by Proposition 4, α is irreducible. Therefore X has a presentation $0 \to L \xrightarrow{\alpha} P \to X \to 0$ with projective module P and irreducible morphism α.

(ii) $\Rightarrow$ (i): Let $0 \to L \xrightarrow{\alpha} P \to X \to 0$ be a presentation of X with projective module P and irreducible morphsim α. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} X \to 0$ be a non split exact sequence and C be a proper submodule of X. Then we have the pull back diagram

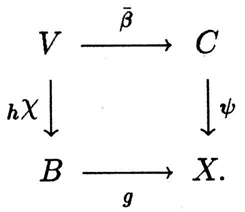
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\bar{f}} & W & \xrightarrow{\bar{g}} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & X & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\bar{\alpha}} & V & \xrightarrow{\bar{\beta}} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & L & \xrightarrow[\alpha]{} & P & \xrightarrow[\beta]{} & X & \longrightarrow & 0, \end{array}$$

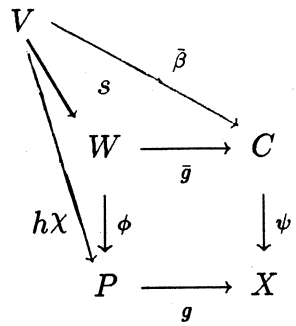
where ψ is the inclusion monomorphism and α is the irreducible morphism. Therefore β can not be lifted to C because ψ is an inclusion monomorphsim. Since α is an irreducible morphism, β can be lifted to P by Proposition 4. So there is $\theta : C \to P$ such that $\beta\theta = \psi$. By the universal property of pull backs:



the sequence $0 \to L \to V \to C \to 0$ splits. Since P is projective, the map $\beta : P \to X$ can be lifted to B. Hence there is a map $h : P \to B$ such that $gh = \beta$ and we have a commutative diagram:



By the universal property of pull backs, there is a unique map s : $V \to W$ such that the diagram



commutes. Since the map $\bar{\beta}$ splits, the map $\bar{g}$ splits. Therefore X is an almost projective module. This completes the proof.

The following result gives some important characterization of the almost projective modules.

THEOREM 6. *Every almost projective module is indecomposable.*

PROOF. Suppose that an almost projective module X is decomposable. Assume that $X = X_1 \oplus X_2$, $X_1 \neq 0$, $X_2 \neq 0$. Then by Theorem 5, there is a presentation of X, $0 \to L \xrightarrow{\alpha} P \xrightarrow{\beta} X \to 0$ with projective module P and irreducible morphism α. Given canonical monomorphisms $i_i : X_i \to X$, $i = 1, 2$, β can not be lifted to X_i because i_i is a monomorphism. Since α is a irreducible morphism, by Proposition 4, i_i can be lifted to P. Hence there is a map $h_i : X_i \to P$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} & & & & & & X_i & & \\ & & & & & \swarrow^{h_i} & \downarrow{\scriptstyle i_i} & & \\ 0 & \longrightarrow & L & \xrightarrow[\alpha]{} & P & \xrightarrow[\beta]{} & X & \longrightarrow & 0. \end{array}$$

Since i_i is a monomorphism, h_i is also a monomorphism. Let $\pi_i : X \to X_i$, $i = 1, 2$, be canonical epimorphisms. Then $\pi_i \beta h_i = \pi_1 i_i = 1_{X_i}$. Hence $\pi_i \beta : P \to X_i$ splits h_i. Therefore $h_i(X_i)$ is a summand of P. Since P is projective and $X_i \cong h_i(X_i)$, each X_i, $i = 1, 2$, is projective. Hence $X = X_1 \oplus X_2$ is projective. This contradicts to the assumption that X is an almost projective module. Therefore X is indecomposable. This completes the proof.

Now we only consider modules in mod Λ. As we know, every modules in mod Λ is artinian and noetherian. Since an almost projective module X is indecomposable, artinian and noetherian Λ-module, $E_\Lambda(X)$ = the set of all Λ-homomorphisms from X into X is a local algebra. Hence every cyclic $E_\Lambda(X)$-module is indecomposable [4].

The next propositions will be used to prove the Theorem 9.

PROPOSITION 7. *Almost projective modules are of bounded length.*

PROOF. See Corollary 1 and Proposition 9 in [7].

PROPOSITION 8. *Let* $Q_m \xrightarrow{\chi_m} Q_{m-1} \xrightarrow{\chi_{m-1}} Q_{m-2} \to \cdots Q_1 \xrightarrow{\chi_1} Q_0$ *be a sequence of homomorphisms of finitely generated indecomposable modules. Assume that* $\ell(Q_i) \le n$ *and none of the* χ_i *is an isomorphism. If* $m \ge 2^n$ *then* $\chi_1\chi_2\cdots\chi_m = 0$.

PROOF: See p.112 in [4].

THEOREM 9. *If* $X_m \xrightarrow{\chi_m} X_{m-1} \xrightarrow{\chi_{m-1}} X_{m-2} \to \cdots \to X_1 \xrightarrow{\chi_1} X_0$ *is a sequence of homomorphisms of almost projective modules such that* χ_i, $1 \le i \le m$, *is not an isomorphism, then* $m \ge 2^n$ *implies* $\chi_1\chi_2\cdots\chi_m = 0$ *for some* n.

PROOF. Since every almost projective module is of bounded length by Proposition 7, $\ell(X_i) \le n$ for some n. By Theorem 6, every almost projective module is indecomposable. Hence $m \ge 2^n$ implies $\chi_1\chi_2\ldots\chi_m = 0$ by Proposition 8.

We will consider the algebras which every proper submodule of an almost projective module is projective.

THEOREM 10. *Let* Λ *be an hereditary algebra. Then every proper submodule of an almost projective module is projective.*

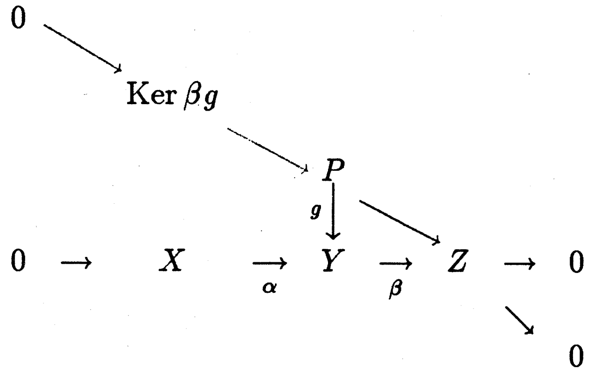
PROOF. Let X be an almost projective module and C be a proper submodule of X. Since X is almost projective, by Theorem 5, there is a presentation of X, $0 \to L \xrightarrow{\alpha} P \to X \to 0$ with P projective and α irredeucible. Let $i : C \to X$ be an inclusion monomorphism. Since α is irreducible, i can be lifted to P. So there is $h : C \to P$ such that $\beta h = i$. Therefore h is a monomorphism, and so $C \cong h(C) \subseteq P$. Since P is a projective module and every submodule of projective module is projective on hereditary algebra, C also is projective.

The next result is characterization of almost projective modules by means of irreducible morphisms.

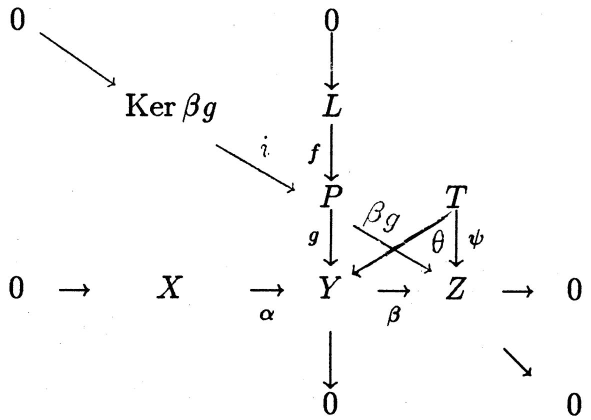
THEOREM 11. *If* $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ *is an exact sequence, where* Y *is almost projective and* α *is irreducible, then* Z *is also almost projective.*

PROOF. Since Y is almost projective, by Theorem 5, there is a presentation $0 \to L \xrightarrow{f} P \xrightarrow{g} Y \to 0$ with P projective and f irreducible.

Hence the composition $\beta g : P \to Z$ is an epimorphism. Then there is an exact sequence $0 \to \operatorname{Ker}\beta g \xrightarrow{\beta g} Z \to 0$, where $i : \operatorname{Ker}\beta g \to P$ is an inclusion morphism:



To show that i is irreducible, we choose any morphism $\psi : T \to Z$. We will use the irreducibility of α and f to show that i is irreducible. Since β is an epimorphism, β can not be lifted to T. Since α irreducible, there is a morphism $\theta : T \to Y$ such that the following diagram commutes:

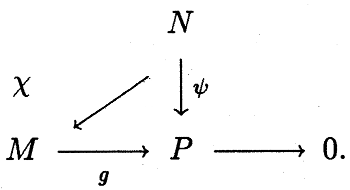


Since f is an irreducible morphism, either there is a morphism ϕ : $T \to P$ such that $g\phi = \theta$ or there is a morphism $\phi : P \to T$ such that $\theta h = g$. Then $\beta g \phi = \beta\theta = \psi$ or $\psi h = \beta\theta h = \beta g$. Therefore Z is almost projective by Theorem 5.

M. Auslander and I. Reiten gave some alternative characterizations of almost split sequences and made considerable progress toward an understanding of their structure. Basic references of these sequences are the papers, [**1**], [**2**] and [**4**].

DEFINITIONS 12. An almost split sequence is a short exact sequence $\sum : 0 \to Q \to M \xrightarrow{g} P \to 0$ such that Q and P are indecomposable modules and $\sum$ is not split and if $\psi : N \to P$ is a module homomorphism that is not split surjective then ψ factors through g, that is $\psi = gX$ for some $X : N \to M$.

The last condition can be expressed as a diagram that is analogous to the characterization of projectivity:



We now state consequence for almost split sequences in mod Λ.

LEMMA 13. *Let* $0 \to Q \xrightarrow{\psi} M \xrightarrow{\phi} P \to 0$ *be an almost split sequence. Suppose that* $\theta : Q \to N$ *is not a split injection. Then there exists* $X : M \to N$ *such that* $X\psi = \theta$.

PROOF. See p,119 in [**4**].

If Λ is an artin algebra over the commutative artin ring R, we will set D denote the functor Hom($, I_0(R/\operatorname{Rad} R))$ from mod Λ to mod Λ^{op}, where Λ^{op} is the opposite ring of Λ, I_0 stands for the injective envelope and $\operatorname{rad} R$ is the radical of R.

Let M be a module in mod Λ and $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a minimal projective presentation of M. We now apply the functor $\operatorname{Hom}_\Lambda(\ , \Lambda)$ the above presentation and the cokernel of $\operatorname{Hom}_\Lambda(f, \ )$ is denoted by trM.

PROPOSITION 14(AUSLANDER AND REITEN, [**1**]. *If X is a finitely generated non projective indecomposable module then there is a unique up to isomorphism, almost split sequence* $0 \to A \to B \to X \to 0$. *Moreover* $A \cong \operatorname{Dtr}(X)$.

COROLLARY 15. *For an almost projective module X, there is an almost split sequence* $0 \to \operatorname{Dtr}(X) \to E \to X \to 0$.

PROOF: If we apply Theorem 6 and Proposition 14 then the above result holds.

The following theorem is a characterization of almost projective modules by means of almost split sequences.

THEOREM 16. *Let X be an injective almost projective non simple module and* $0 \longrightarrow \mathrm{Dtr}(X) \longrightarrow E \longrightarrow X \longrightarrow 0$ *be an almost split sequence. If* $E = E_1 \oplus E_2$ *where* E_1 *and* E_2 *are indecomposable, then* $\ell(X) = 3$.

PROOF. Since X is an injective indecomposable non projective, non simple, $0 \longrightarrow \mathrm{Dtr}(X) \longrightarrow E \longrightarrow X \longrightarrow 0$ is an almost split sequence and E is not indecomposable, E is projective ([**7**, Theorem 13]). Therefore summands E_1, E_2 of E are projective modules. Then the irreducible maps $E_i \longrightarrow X$, $i = 1, 2$, are monomorphisms and $\mathrm{Dtr}(X) \cong rE \cong rE_1$ where rE is the radical of E and $\ell(E_1) = \ell(E_2) = 2$ ([**5**, Corollary 1, 2]). Since E is projective,$\mathrm{Dtr}(X)$ is simple ([**1**], Proposition 5,5). Since E is projective, $\mathrm{Dtr}(X)$, E and X are artinian and noetherian modules, $\ell(E) = \ell(\mathrm{Dtr}(X)) + \ell(X)$. Therefore we have $\ell(X) = 3$.

We will consider modules over an algebra which is an almost projective modules.

PROPOSITION 17. *If* Λ *is artinian then,* Λ *is hereditary and* $(\mathrm{Rad}\,\Lambda)^2 = 0$ *if and only if every simple* Λ*-moudle is either projective or injective.*

PROOF. See [**3**] and [**8**].

Rings for which each submodule of a quasi-projective module is quasi-projective were called the completely hereditary ring by Fuller. And he proved that an artinian ring R is completely hereditary if and only if each simple module is either injective or projective. Since every simple module is injective over a commutative Von Neumann regular ring, a commutative Von Neumann regular ring is completely hereditary.

THEOREM 18. *If* Λ *is artinian and completely hereditary then every simple injective module is an almost projective module.*

PROOF. Every simple injective module can not be projective over Λ. Since the simple non projective module is almost projective, every simple injective module is an almost projective module.

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